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KNOTTED ATTRACTING PERIODIC ORBITS IN SILNIKOV BIFURCATIONS

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1. INTRODUCTION

Let X be a C^1-vector field on $\mathbb{R}^3$ which satisfies the following conditions;

(i) $X(0)=0$,

(ii) $DX(0)$ has eigenvalues $\lambda, -\mu\pm i\omega$ with $0<\mu<\lambda, \omega\neq 0$,

(iii) $W^s(0)$ and $W^u(0)$ intersect at a homoclinic orbit Γ.

We call such a vector field a Silnikov system. A Silnikov system has infinitely many periodic orbits arbitrally close to the homoclinic orbit Γ[3]. Moreover the knot types of these periodic orbits and their linking numbers around Γ determines μ/λ. Therefore μ/λ is a modulus, that is , if X and X' are two Silnikov systems with $\mu/\lambda\neq\mu'/\lambda'$, then they are not topologically equivalent.[4][5]. If a Silnikov system X has eigenvalues $\lambda, -\mu\pm i\omega$ with $0<\mu<\lambda<2\mu$ and $\omega\neq 0$, then one can perturb X to X' so that X' has an attracting periodic orbit arbitrally close to Γ. Attracting periodic orbits in $\mathbb{R}^3$ are the simplest attractors which can have non trivial embedding types. So it would be important to ask whether μ/λ gives any restrictrions on the possible embedding types of attracting periodic orbits of perturbed systems. In this paper, we prove that:

THEOREM *Let* X *and* X' *be Silnikov systems with homoclinic orbits* Γ *and* Γ' *and eiginvalues* $\lambda, -\mu \pm i\omega$ *and* $\lambda', -\mu' \pm i\omega'$ *respectively. Suppose that* $0<\mu<\lambda<2\mu$ *and* $\mu/\lambda<\mu'/\lambda'$.
Let X_s *be a 1-parameter family of vector fields with* $X_0=X$. *Let* X_s *satisfies the transversality condition defined later. Then there exists a* $\delta>0$ *such that for any* $\varepsilon>0$ *there exists a positive* $s<\varepsilon$ *such that*

(i) X_s *has an attracting periodic orbit* γ *in the* ε*-neighbourhood of* Γ, *and*

(ii) *any* δ*-perturbation* X'' *of* X' *has no periodic orbit in the* δ*-neighbourhood of* Γ' *whose embedding type into the* δ*-neighbourhood is the same of that of* γ.

To prove the theorem we use the attracting periodic orbits which trips three times arong Γ.

2. ANALYTIC STUDY OF ATTRACTING PERIODIC ORBITS

Let S be the unit cylinder;

$$S=\{(x_1,x_2,x_3) \mid x_1{}^2+x_2{}^2\le 1,\ |x_3|\le 1\}.$$

We use (r,θ,z)-coordinate in the unit cylinder S;

$$x_1=r\cos\theta,\ x_2=r\sin\theta,\ x_3=z.$$

Let SS be the side surface of S;

$$SS=\{(x_1,x_2,x_3) \mid x_1{}^2+x_2{}^2=1,\ |x_3|\le 1\}.$$

Let X_s be a 1-parameter family of C^1-vector fields on $\mathbb{R}^3$ with X_0 Silnikov. To avoide the technical diffical ties, we suppose the following;

(*) there exists a 1-parameter family of diffeomorphisms which linearlizes X_s in the unit cylinder.

Take a sufficiently small r_1 so that any X_s which starts at a point p in

$$\Sigma^2=\{(r,\theta,1)|\ r\leq r_1\}$$

crosses SS at a point $f_s(p)$, and the mapping

$$f_s: p \in \Sigma^2 \dashrightarrow f_s(p) \in SS$$

is a diffeomorphism onto its image. For each s let hight(s) denote the z-coordinate of the point at which $W^u(0)$ crosses SS. We suppose the transversality condition;

$$\frac{d}{dt}\text{hight}(s)|_{s=0} \neq 0.$$

Let L_s be the map SS ---> $\{(r,\theta,1)|\ r\leq 1\}$ induced by the linear vector field $X_s|S$. Let AP(s,ε) be the set of attracting periodic orbit γ of X_s which satisfies

(i) γ is in the ε-neighbourhood of Γ,

(ii) γ crosses SS at three points, say $P_1(\gamma)$, $P_2(\gamma)$ and $P_2(\gamma)$ with z-coordinates $m_1(\gamma)$, $m_2(\gamma)$ and $m_3(\gamma)$,

(iii) $f_s \circ L_s(p_i)=P_{i+1}$ (mod 3) and $m_1(\gamma)<m_2(\gamma)<m_3(\gamma)$.

Note that AP(s,ε) may be empty. Let

$$\eta(s,\varepsilon)=\inf\frac{\log m_3(\gamma)}{\log m_2(\gamma)}$$

where infimum is taken over all γ's in AP(s',ε) with $0<s'\leq s$.

PROPOSITION 1. *For any* $\varepsilon_0>0$ *there exists a* $\delta>0$ *such that*

$$\eta(s,\delta)>\mu/\lambda - \varepsilon_0$$

for all $|s|<\delta$.

PROPOSITION 2. $\eta(S_0,\varepsilon_0)\leq\mu/\lambda$ *for sufficiently small* $s_0>0$ *and* ε_0.

2. PROOF OF THE PROPOSITIONS.

Proof of proposition 1. Let $c(s)=\mu/\lambda$. If $\delta_0>0$ is sufficiently small, then there exists a constant K such that

$$\text{hight}(s) - K(m_1(\gamma))^{c(s)} \leq m_2(\gamma),$$
$$m_3(\gamma) \leq \text{hight}(s)+K(m_2(\gamma))^{c(s)}$$

for all γ in $AP(s,\delta_0)$ with $0<s\leq\delta_0$. Since $m_1(\gamma)<m_2(\gamma)$, we get

$$m_3(\gamma)\leq(1+2K)(m_2(\gamma))^{c(s)}.$$

Taking the logarizm, we get

$$\frac{\log m_3(\gamma)}{\log m_2(\gamma)} \geq c(s) + \frac{1+2K}{\log m_2(\gamma)} .$$

Taking a smaller δ, we get the required inequality.

Proof of proposition 2. The proposition 2 would be prooved by a standard but very messy arguments.

3. TOPOLOGICAL INVARIANT

The braid of the periodic orbit in $AP(s,\varepsilon)$ is indicated in the fig 1, where β is an unknown but fixed braid. The numbers N_1 and N_2 goes to infinity as ε goes to zero. The ratio of N_1 and N_2 is estimated by the propositions 1 and 2, so the remaining task is to relate the numbers N_1 and N_2 to some topological invariants. Hence the following proposition completes the proof of the theorem.

Let $\Delta_\gamma(t)$ be the Alexander polynomial of the knot γ^2.) Let $n(\gamma)$ be the number of the non zero terms of the polynomial $\Delta_\gamma(t)$.

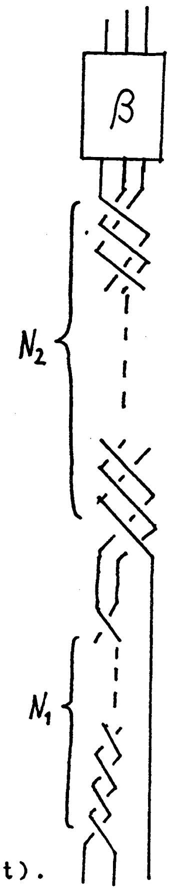

Fig 1

PROPOSITION 3. *There exists a constant* C *such that*

$$|\ m(t)-N_2\ | < C$$

$$|\ \text{degree of } \Delta_\gamma(t) - (6N_1+N_2)\ | < C,$$

for all N_1 *and* N_2.

Proof. Let $\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ be the Peron Mugnus representaion of the braid $\beta^{1)}$. Then the Alexander polynomial $\Delta_\gamma(t)$ is

$$(-1)^m t^{-(6n+m)}(a(t)d(t)-b(t)c(t)) + t^{-3n}b(t)\sum_{K=1}^{m}(-t)^{-k}$$
$$- (-1)^m t^{-(3n+m)}a(t) - t^{-3n}d(t) + 1,$$

where $n=N_1$ and $m=N_2$. Now the proposition is clear, since neither b(t) nor a(t)d(t)-b(t)c(t) is a zero polynomial.(Note that neither a(1)d(1)-b(1)c(1) nor b(-1) is zero).

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